

Optimal super dense coding over noisy quantum channels

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Abstract. We investigate super dense coding in the presence of noise, i.e., the subsystems of the entangled resource state have to pass a noisy unital quantum channel between the sender and the receiver. We discuss explicitly the case of Pauli channels in arbitrary dimension and derive the super dense coding capacity (i.e. the optimal information transfer) for some given resource states. For the qubit depolarizing channel, we also optimize the super dense coding capacity with respect to the input state. We show that below a threshold value of the noise parameter the super dense coding protocol is optimized by a maximally entangled initial state, while above the threshold it is optimized by a product state. Finally, we provide an example of a noisy channel where non-unitary pre-processing increases the super dense coding capacity, as compared to only unitary encoding.

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1. Introduction

In quantum information processing, entanglement can be used as a resource for super dense coding, as introduced by Bennett et al. [1]. Essential to this communication protocol is an entangled initial state that is shared between sender(s) and receiver(s), together with the property that an entangled state can be transformed by the sender into another state via a *local* operation, taken from some set of operations. The sender's subsystem is then transmitted to the receiver (ideally via a noiseless channel), who identifies the global state in an optimal way. The super dense coding capacity is defined to be the maximal amount of classical information that can be reliably transmitted to the receiver for a given initial state. In the last years attention has been given to various scenarios of super dense coding over noiseless channels [2, 3, 4]. It has been proved that for noiseless channels and for unitary encoding, the super dense coding capacity is given by [2]

$$C = \log d + S(\rho_b) - S(\rho) , \quad (1)$$

where ρ is the initial resource state shared between the sender (Alice) and the receiver (Bob). Here, d is the dimension of Alice's system, ρ_b is Bob's reduced density operator and $S(\rho) = -\text{tr}(\rho \log \rho)$ is the von Neumann entropy. Without the additional resource of entangled states, a d -dimensional quantum state can be used to transmit the information $\log d$. Hence, quantum states for which $S(\rho_b) - S(\rho) > 0$ are the states which are useful for dense coding. The relation $S(\rho_b) - S(\rho) > 0$ cannot hold for quantum states with positive partial transpose [3]. Therefore, states that are useful for dense coding always have a non-positive partial transpose (NPT). However, the converse is not true: There exist states which are NPT but which are not useful for dense coding. One can then classify bipartite states according to their usefulness for super dense coding [4]. Besides the case of a single sender and receiver sharing an initial pure entangled state and using unitary encoding some other scenarios also have been discussed: many senders and either one or two receivers, initially entangled mixed states, non-unitary encoding, etc. [1, 2, 4, 5]. Super dense coding has been realized in optical experiments with polarized photons by Mattle et al. [6], and for continuous variables by Li *et al* [7].

In a realistic scenario however noise is unavoidably present. The central theme of this paper is the question: how does noise in the transmission channel affect the superdense coding capacity? Here, we focus on the case of a single sender and a single receiver, assuming unitary encoding at first, and then generalizing to non-unitary encoding. Physically, noise is a process that arises through interaction with the environment. Mathematically, a noisy quantum channel can be described as a completely positive trace preserving linear map Λ , acting on the quantum state. In this paper we will study two different scenarios of noisy channels: first, we will assume that the sender Alice and the receiver Bob share already a bipartite quantum state ρ (it could e.g. have been distributed to them by a third party). After Alice's local encoding operation, she sends her part of the quantum state to Bob via the noisy channel, described by the map Λ_a , see Figure 1. We call this the case of a *one-sided* channel.

Second, we consider the case where Alice prepares the bipartite state ρ and sends one part of it via a noisy channel, described by the map Λ_b , to Bob, thus establishing the shared resource state for super dense coding. When the two parties want to use this resource, Alice does the local encoding and then sends her part of the state via the channel Λ_a to Bob, see Figure 2. We call this case a *two-sided* channel.

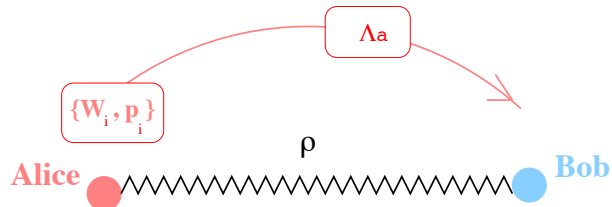


Figure 1. *One-sided* noise: Bipartite super dense coding with an initially entangled state ρ , shared between Alice and Bob. Alice applies the unitary operator W_i , taken from a set $\{W_i\}$ with probability $\{p_i\}$, on her part of the entangled state ρ . She sends the encoded state with probability p_i over a noisy channel, described by the map Λ_a , to Bob. In the first approach we assume that Λ_a just affects Alice's subsystem, but that there is no noise on Bob's side.

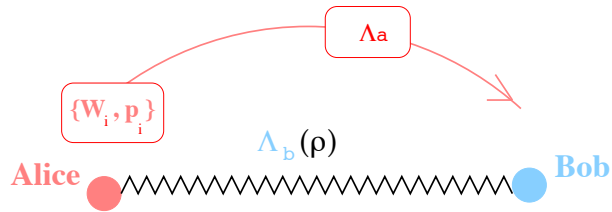


Figure 2. *Two-sided* noise: Bipartite super dense coding with an initially entangled state ρ , shared between Alice and Bob. In the second approach, the noisy channel Λ_a influences Alice's subsystem after encoding while the noisy channel Λ_b has already affected Bob's side in the distribution step of the initial state ρ .

The paper is organized as follows: in Section II we discuss the definition of the Holevo quantity for an ensemble of states in the presence of a noisy channel. We introduce a certain condition on the von Neumann entropy and we derive the super dense coding capacity for those cases where this condition is fulfilled. In Sections III and IV, we give examples of initial states and channels for which this condition on the von Neumann entropy is satisfied, and calculate their optimal super dense coding capacity explicitly. Section V provides a comparison between the super dense coding capacities in the presence of a *one-sided* or *two-sided* 2-dimensional depolarizing channel, and the classical capacity of a 2-dimensional depolarizing channel. In Section VI we consider the case of non-unitary encoding and show an example where pre-processing is useful to increase the dense coding capacity of the initial resource state in the presence of the noisy channel.

2. Super dense coding capacity

In the super dense coding protocol Alice and Bob share a bipartite entangled quantum state ρ . Alice performs local unitary operations W_i with probability p_i (where $\sum_i p_i = 1$) on ρ to encode classical information through the state ρ_i , i.e.

$$\rho_i = (W_i \otimes \mathbb{1})\rho(W_i^\dagger \otimes \mathbb{1}). \quad (2)$$

We consider $\Lambda : \rho_i \rightarrow \Lambda(\rho_i)$ to be any completely positive map that acts on the shared state ρ_i . (Below Λ will describe the noise acting on the ensemble states.) The ensemble that Bob(s) receives is $\{\Lambda(p_i, \rho_i)\}$. The amount of classical information transmitted via a quantum channel is measured by the Holevo quantity or χ -quantity. This quantity for the ensemble $\{\Lambda(p_i, \rho_i)\}$ is given by

$$\chi = S\left(\overline{\Lambda(\rho)}\right) - \sum_i p_i S\left(\Lambda(\rho_i)\right) = \sum_i p_i S\left(\Lambda(\rho_i) \parallel \overline{\Lambda(\rho)}\right), \quad (3)$$

where $\overline{\Lambda(\rho)} = \sum_i p_i \Lambda(\rho_i)$ is the average state and $S(\eta)$ is the von Neumann entropy of η . The symbol $S(\sigma \parallel \rho)$ denotes the relative entropy, defined as $S(\sigma \parallel \rho) = \text{tr}(\sigma \log \sigma - \sigma \log \rho)$. Note that χ is a function of the resource state ρ , the encoding $\{p_i, W_i\}$ and the channel Λ . For brevity of notation we will not write explicitly these arguments of χ .

The super dense coding capacity C for a given resource state ρ is defined to be the maximum of the Holevo quantity χ with respect to $\{p_i, W_i\}$, that is

$$C = \max_{\{p_i, W_i\}} (\chi). \quad (4)$$

In this paper we consider bipartite systems, where each subsystem has finite dimension d . A general density matrix on $\mathbb{C}^d \otimes \mathbb{C}^d$ in the Hilbert-Schmidt representation can be conveniently decomposed as

$$\rho = \mathbb{1} \otimes \frac{\rho_b}{d} + \frac{1}{d^2} \left(\sum_{i=1}^{d^2-1} r_i \lambda_i \otimes \mathbb{1} + \sum_{i,k=1}^{d^2-1} t_{ik} \lambda_i \otimes \lambda_k \right), \quad (5)$$

where $\rho_b = \text{tr}_a \rho$ represents Bob's reduced density operator and λ_i are the generators of the $\text{SU}(d)$ algebra with $\text{tr} \lambda_i = 0$. The parameters r_i, s_i, t_{ik} are real numbers. We introduce the set of unitary operators $\{V_i\}$, defined as

$$V_{i=(m,n)} |j\rangle = \exp\left(\frac{2\pi i n j}{d}\right) |j + m \pmod{d}\rangle. \quad (6)$$

These operators satisfy the condition $d^{-1} \text{tr}(V_i V_j^\dagger) = \delta_{ij}$. Integers m and n run from 0 to $d-1$ such that we have d^2 unitary operators V_i . We will consider in the following the case of unital noisy channels acting on Alice's and Bob's systems, namely channels described by the completely positive map

$$\Lambda(\rho) = \sum_m K_m \rho K_m^\dagger, \quad \sum_m K_m^\dagger K_m = \mathbb{1}, \quad \sum_m K_m K_m^\dagger = \mathbb{1}, \quad (7)$$

where K_m are Kraus operators. Here, the first condition on the Kraus operators corresponds to trace preservation, and the second condition guarantees the unital property $\Lambda(\mathbb{1}) = \mathbb{1}$. We will show in this section that for unital memoryless noisy quantum channels and certain initial resource states, the set of unitary operators $\{V_i\}$ with equal probabilities is the optimum encoding and leads to the maximum of the Holevo quantity.

We will first prove in Lemma 1 some properties that hold for the specific encoding $\{V_i\}$. In the following the symbol τ_i will denote the resource state after encoding with V_i , whereas τ will denote the resource state after encoding with an arbitrary unitary operation U . The ensemble average after the specific encoding with $\{V_i\}$, the probability distribution $p_i = 1/d^2$ and after action of the channel will be denoted as $\tilde{\rho}$. - For similar methods in the case of noiseless channels see also [2].

Lemma 1. Let $\Lambda_a(\sigma_a) = \sum_m A_m \sigma_a A_m^\dagger$ and $\Lambda_b(\sigma_b) = \sum_{\tilde{m}} B_{\tilde{m}} \sigma_b B_{\tilde{m}}^\dagger$ be any two unital channels which act on Alice's and Bob's side, respectively. For an initial resource state ρ shared between Alice and Bob, the global channel Λ_{ab} then acts as

$$\Lambda_{ab}(\rho) = \sum_{m, \tilde{m}} (A_m \otimes B_{\tilde{m}}) \rho (A_m^\dagger \otimes B_{\tilde{m}}^\dagger). \quad (8)$$

Then, the following statements hold:

1-a) For $\tau_i = (V_i \otimes \mathbb{1})\rho(V_i^\dagger \otimes \mathbb{1})$, with V_i being defined in (6), the average $\tilde{\rho}$ of the ensemble $\{p_i = \frac{1}{d^2}, \Lambda_{ab}(\tau_i)\}_{i=0}^{d^2-1}$ takes the form $\tilde{\rho} = \mathbb{1} \otimes \Lambda_b(\frac{\rho_b}{d})$.

1-b) For $\tau = (U \otimes \mathbb{1})\rho(U^\dagger \otimes \mathbb{1})$ with U being any unitary operator acting on Alice's system, $\text{tr}(\Lambda_{ab}(\tau) \log \tilde{\rho}) = -S(\tilde{\rho})$.

1-c) The relative entropy between $\Lambda_{ab}(\tau)$ and $\tilde{\rho}$ can be expressed as $S(\Lambda_{ab}(\tau) \parallel \tilde{\rho}) = S(\tilde{\rho}) - S(\Lambda_{ab}(\tau))$.

Proof 1-a). In [2] it was shown that the average of the ensemble $\{p_i = \frac{1}{d^2}, \tau_i\}_{i=0}^{d^2-1}$ is

$$\sum_i \frac{1}{d^2} \tau_i = \mathbb{1} \otimes \frac{\rho_b}{d}. \quad (9)$$

By using (9), the linearity of the channel and its unital property, the average of the ensemble $\{p_i = \frac{1}{d^2}, \Lambda_{ab}(\tau_i)\}_{i=0}^{d^2-1}$ is

$$\tilde{\rho} = \sum_i \frac{1}{d^2} \Lambda_{ab}(\tau_i) = \Lambda_{ab}\left(\frac{\mathbb{1}}{d} \otimes \rho_b\right) = \mathbb{1} \otimes \Lambda_b\left(\frac{\rho_b}{d}\right). \quad (10)$$

Proof 1-b). In Lemma (1-a) we showed that $\tilde{\rho} = \mathbb{1} \otimes \Lambda_b(\frac{\rho_b}{d})$ and hence, $\log \tilde{\rho} = \mathbb{1} \otimes \log \Lambda_b(\frac{\rho_b}{d})$. Therefore:

$$\begin{aligned}
\text{tr}(\Lambda_{ab}(\tau) \log \tilde{\rho}) &= \text{tr} \left[\left(\sum_m A_m U U^\dagger A_m^\dagger \right) \otimes \left(\sum_{\tilde{m}} B_{\tilde{m}} \frac{\rho_b}{d} B_{\tilde{m}}^\dagger \log \Lambda_b \left(\frac{\rho_b}{d} \right) \right) \right. \\
&+ \frac{1}{d^2} \left(\sum_{i=1}^{d^2-1} r_i \sum_m A_m U \lambda_i U^\dagger A_m^\dagger \right) \otimes \left(\sum_{\tilde{m}} B_{\tilde{m}} B_{\tilde{m}}^\dagger \log \Lambda_b \left(\frac{\rho_b}{d} \right) \right) \\
&\left. + \frac{1}{d^2} \sum_{i,k=1}^{d^2-1} t_{ik} \left(\sum_m A_m U \lambda_i U^\dagger A_m^\dagger \right) \otimes \left(\sum_{\tilde{m}} B_{\tilde{m}} \lambda_k B_{\tilde{m}}^\dagger \log \Lambda_b \left(\frac{\rho_b}{d} \right) \right) \right]. \quad (11)
\end{aligned}$$

By using the linearity of the trace and the relations

$$\text{tr} \left[\sum_m A_m U U^\dagger A_m^\dagger \right] = \text{tr} \left[\sum_m A_m A_m^\dagger \right] = \text{tr}[\mathbb{1}], \quad (12)$$

$$\begin{aligned}
\text{tr} \left[\sum_m A_m U \lambda_i U^\dagger A_m^\dagger \right] &= \text{tr} \left[U \lambda_i U^\dagger \sum_m A_m^\dagger A_m \right] \\
&= \text{tr} [U \lambda_i U^\dagger] = \text{tr} [\lambda_i] = 0
\end{aligned} \quad (13)$$

we can write

$$\begin{aligned}
\text{tr}(\Lambda_{ab}(\tau) \log \tilde{\rho}) &= \text{tr}_a \text{tr}_b \left[\sum_{m,\tilde{m}} \mathbb{1} \otimes \left(B_{\tilde{m}} \frac{\rho_b}{d} B_{\tilde{m}}^\dagger \log \Lambda_b \left(\frac{\rho_b}{d} \right) \right) \right] \\
&= \text{tr}_b \left[\Lambda_b(\rho_b) \log \Lambda_b \left(\frac{\rho_b}{d} \right) \right] = -S(\tilde{\rho}).
\end{aligned} \quad (14)$$

Proof 1-c). Using the definition of the relative entropy $S(\sigma\|\rho) = \text{tr}(\sigma \log \sigma - \sigma \log \rho)$ and the result of Lemma (1-b) we can write

$$\begin{aligned}
S(\Lambda_{ab}(\tau) \|\tilde{\rho}) &= \text{tr}(\Lambda_{ab}(\tau) \log \Lambda_{ab}(\tau) - \Lambda_{ab}(\tau) \log \tilde{\rho}) \\
&= S(\tilde{\rho}) - S(\Lambda_{ab}(\tau)).
\end{aligned} \quad (15)$$

□

We now show that for resource states with a certain symmetry property, namely for those states where the von Neumann entropy after the channel action is independent of the unitary encoding, the encoding with the equally probable operators $\{V_i\}$, as given in (6), is optimal. Our proof follows the line of argument developed in [2].

Lemma 2. Let τ_i denote the resource state after encoding with V_i , given in (6). Let

$$\tilde{\chi} = S(\tilde{\rho}) - \frac{1}{d^2} \sum_i^{d^2-1} S(\Lambda_{ab}(\tau_i)) \quad (16)$$

be the Holevo quantity for the ensemble $\{p_i = \frac{1}{d^2}, \Lambda_{ab}(\tau_i)\}$, where $\tilde{\rho}$ is the average state of this ensemble and $\Lambda_{ab}(\cdot)$ is defined in (8). For all the channels Λ_{ab} and all initial states ρ for which

$$S(\Lambda_{ab}(\tau)) = \frac{1}{d^2} \sum_i^{d^2-1} S(\Lambda_{ab}(\tau_i)) \quad (17)$$

holds, $\tilde{\chi}$ is the super dense coding capacity. Here $\tau = (U \otimes \mathbb{1}) \rho (U^\dagger \otimes \mathbb{1})$, as we defined already above, with U being any unitary operator.

Proof. Let us consider an arbitrary encoding, leading to an ensemble $\{p_i, \Lambda_{ab}(\rho_i)\}$. We will show that its Holevo quantity χ cannot be higher than $\tilde{\chi}$ in (16), if the condition (17) is fulfilled.

If $S(\Lambda_{ab}(\tau)) = \frac{1}{d^2} \sum_i^{d^2-1} S(\Lambda_{ab}(\tau_i))$, then from (16) and Lemma (1-c),

$$\tilde{\chi} = S(\Lambda_{ab}(\tau) \| \tilde{\rho}). \quad (18)$$

Since this equation holds for any τ that fulfills (17), it specially holds for ρ_i , i.e.

$$\tilde{\chi} = S(\Lambda_{ab}(\rho_i) \| \tilde{\rho}) = \sum_i p_i S(\Lambda_{ab}(\rho_i) \| \tilde{\rho}). \quad (19)$$

Using Donald's identity, see [8], the right hand side of the above equation can be decomposed as

$$\sum_i p_i S(\Lambda_{ab}(\rho_i) \| \tilde{\rho}) = \sum_i p_i S(\Lambda_{ab}(\rho_i) \| \overline{\Lambda_{ab}(\rho)}) + S(\overline{\Lambda_{ab}(\rho)} \| \tilde{\rho}) \quad (20)$$

with $\overline{\Lambda_{ab}(\rho)} = \sum_i p_i \Lambda_{ab}(\rho_i)$. The first term on the right hand side is the Holevo quantity for any arbitrary ensemble $\{p_i, \Lambda_{ab}(\rho_i)\}$. Hence,

$$\tilde{\chi} = \chi + S(\overline{\Lambda_{ab}(\rho)} \| \tilde{\rho}). \quad (21)$$

Since the relative entropy $S(\overline{\Lambda_{ab}(\rho)} \| \tilde{\rho})$ is always positive or zero we can say that $\tilde{\chi}$ is always bigger or equal than χ and hence, $\tilde{\chi}$ is the super dense coding capacity. \square

From Lemma 2 we find that

$$\tilde{\chi} = S(\tilde{\rho}) - S(\Lambda_{ab}(\tau)). \quad (22)$$

Since the above equation holds for $\tau = (U \otimes \mathbb{1}) \rho (U^\dagger \otimes \mathbb{1})$ with any unitary U , it especially holds for $\tau = \rho$. Hence, whenever the condition (17) is true, the super dense coding capacity is given by

$$C = \tilde{\chi} = S(\tilde{\rho}) - S(\Lambda_{ab}(\rho)), \quad (23)$$

where $\tilde{\rho}$ is the average of the ensemble after encoding with the specific (and equally probable) unitaries $\{V_i\}$ and after the channel action, as introduced in Lemma 1. As an interpretation of this formula, note that the action of a noisy channel typically will increase the entropy of a given state, and therefore will decrease the dense coding capacity of the original resource state.

In the next two sections we will study examples of channels and bipartite states satisfying the condition (17), and evaluate explicitly the corresponding super dense coding capacities.

3. *One-sided* d -dimensional Pauli channel

A d -dimensional Pauli channel [9] that acts just on Alice's side is defined by

$$\Lambda_a^P(\rho_i) = \sum_{m,n=0}^{d-1} q_{mn} (V_{mn} \otimes \mathbb{1}) \rho_i (V_{mn}^\dagger \otimes \mathbb{1}), \quad (24)$$

where q_{mn} are probabilities (i.e. $q_{mn} \geq 0$ and $\sum_{mn} q_{mn} = 1$). The operators V_{mn} , defined in (6) with a slightly different notation for the indices, can be expressed as

$$V_{mn} = \sum_{k=0}^{d-1} \exp\left(\frac{2i\pi kn}{d}\right) |k\rangle \langle k+m(\bmod d)|. \quad (25)$$

They satisfy $\text{tr } V_{mn} = d\delta_{m0}\delta_{n0}$ and $V_{mn}V_{mn}^\dagger = \mathbb{1}$, and have the properties

$$V_{mn}V_{\tilde{m}\tilde{n}} = \exp\left(\frac{2i\pi\tilde{n}m}{d}\right) V_{m+\tilde{m}(\bmod d), n+\tilde{n}(\bmod d)}, \quad (26)$$

$$\text{tr}[V_{mn}V_{\tilde{m}\tilde{n}}^\dagger] = d\delta_{m\tilde{m}}\delta_{n\tilde{n}}, \quad (27)$$

$$V_{mn}V_{\tilde{m}\tilde{n}} = \exp\left(\frac{2i\pi(\tilde{n}m - n\tilde{m})}{d}\right) V_{\tilde{m}\tilde{n}}V_{mn}. \quad (28)$$

As the Kraus operators of one-sided Pauli channel (24) are unitary it is a unital channel.

3.1. Bell states

A Bell state in $d \times d$ dimensions is defined as $|\psi_{00}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle \otimes |j\rangle$. The set of the other maximally entangled Bell states is then denoted by $|\psi_{mn}\rangle = (V_{mn} \otimes \mathbb{1})|\psi_{00}\rangle$, for $m, n = 0, 1, \dots, d-1$. We will show that for a Bell state shared between Alice and Bob, and with a *one-sided* d -dimensional Pauli channel, the condition (17) is fulfilled. We will first prove the following Lemma.

Lemma 3. Let us define $\pi_{mn} := (V_{mn}U \otimes \mathbb{1})\rho_{00}(U^\dagger V_{mn}^\dagger \otimes \mathbb{1})$, where U is a unitary operator, $\rho_{00} = |\psi_{00}\rangle\langle\psi_{00}|$ and V_{mn} is defined in (25). For $m \neq \tilde{m}, n \neq \tilde{n}$,

$$\pi_{mn}\pi_{\tilde{m}\tilde{n}} = 0 \quad (29)$$

holds.

Proof.

In Appendix B we show that $\rho_{00}(U^\dagger V_{mn}^\dagger V_{\tilde{m}\tilde{n}} U \otimes \mathbb{1})\rho_{00} = 0$ for $m \neq \tilde{m}, n \neq \tilde{n}$. Hence,

$$\pi_{mn}\pi_{\tilde{m}\tilde{n}} = (V_{mn}U \otimes \mathbb{1}) \underbrace{\rho_{00}(U^\dagger V_{mn}^\dagger V_{\tilde{m}\tilde{n}} U \otimes \mathbb{1})\rho_{00}}_0 (U^\dagger V_{\tilde{m}\tilde{n}}^\dagger \otimes \mathbb{1}) = 0$$

□

By using the orthogonality property (29) and the purity of the density operators π_{mn} , we can write

$$\begin{aligned}
S(\Lambda_a^P(\tau)) &= S(\Lambda_a^P((U \otimes \mathbb{1})\rho_{00}(U^\dagger \otimes \mathbb{1}))) \\
&= S\left(\sum_{m,n=0}^{d-1} q_{mn} \underbrace{(V_{mn}U \otimes \mathbb{1})\rho_{00}(U^\dagger V_{mn}^\dagger \otimes \mathbb{1})}_{:=\pi_{mn}}\right) \\
&= H(\{q_{mn}\}), \tag{30}
\end{aligned}$$

where $H(\{q_{mn}\}) = -\sum_{m,n} q_{mn} \log q_{mn}$ is the Shannon entropy. We note that the von Neumann entropy $S(\Lambda_a^P(\tau))$ is independent of the unitary encoding U . Consequently, for a *one-sided* d -dimensional Pauli channel with an initial Bell state, the condition (17) is satisfied. The super dense coding capacity (23) for an initial Bell state and a one-sided Pauli channel in d dimensions takes the form

$$C_{\text{Bell}}^{\text{one-sided P}_d} = S\left(\frac{\mathbb{1}}{d} \otimes \rho_b\right) - H(\{q_{mn}\}) = \log d^2 - H(\{q_{mn}\}) \tag{31}$$

for $m, n = 0, 1, \dots, d-1$. Using (1) we notice that the super dense coding capacity of a $d \times d$ -dimensional Bell state in the noiseless case is given by $\log d^2$. Thus, in the presence of a one-sided Pauli channel the super dense coding capacity is reduced by the amount $H(\{q_{mn}\})$ with respect to the noiseless case - i.e. the channel noise is simply subtracted from the super dense coding capacity with noiseless channels.

Notice that the same capacity is achieved also for any maximally entangled state, i.e. for any $|\psi\rangle = U_a \otimes U_b |\psi_{00}\rangle$. Actually, Lemma 3 still holds in this case and therefore also the derivation of the capacity (31).

3.2. Werner states

We will now evaluate the super dense coding capacity for an input Werner state $\rho_W = \frac{1-\eta}{d^2} \mathbb{1} + \eta \rho_{00}$ with $0 \leq \eta \leq 1$. The Werner state ρ_W in the presence of a one-sided d -dimensional Pauli channel provides another example of states and channels that satisfy (17).

Using (30), $\{q_{mn}\}$ is the set of eigenvalues of $\Lambda_a^P[(U \otimes \mathbb{1})\rho_{00}(U^\dagger \otimes \mathbb{1})]$. The Pauli channel is a linear and unital map. Expressing the identity matrix $\mathbb{1}$ in a suitable basis, we arrive at

$$\begin{aligned}
S(\Lambda_a^P((U \otimes \mathbb{1})\rho_W(U^\dagger \otimes \mathbb{1}))) &= S\left(\eta \Lambda_a^P[(U \otimes \mathbb{1})\rho_{00}(U^\dagger \otimes \mathbb{1})] + \frac{1-\eta}{d^2} \mathbb{1}\right) \\
&= S\left(\text{diag}\left(\eta q_{00} + \frac{1-\eta}{d^2}, \dots, \eta q_{d-1,d-1} + \frac{1-\eta}{d^2}\right)\right) \\
&= H\left(\left\{\eta q_{mn} + \frac{1-\eta}{d^2}\right\}\right). \tag{32}
\end{aligned}$$

From (32) it is apparent that the output channel entropy is independent of the unitary encoding. Consequently, the super dense coding capacity, according to (23), is given by

$$C_{\text{Werner}}^{\text{one-sided } P_d} = \log d^2 - H\left(\left\{\frac{1-\eta}{d^2} + \eta q_{mn}\right\}\right). \quad (33)$$

The above capacity is also achieved by any other state with the form $U_a \otimes U_b \rho_W U_a^\dagger \otimes U_b^\dagger$.

4. Two-sided d -dimensional depolarizing channel.

In (24) we introduced the concept of a *one-sided* d -dimensional Pauli channel. A *two-sided* d -dimensional Pauli channel is then defined by

$$\Lambda_{ab}^P(\rho_i) = \sum_{m,n,\tilde{m},\tilde{n}=0}^{d-1} q_{mn} q_{\tilde{m}\tilde{n}} (V_{mn} \otimes V_{\tilde{m}\tilde{n}}) \rho_i (V_{mn}^\dagger \otimes V_{\tilde{m}\tilde{n}}^\dagger). \quad (34)$$

The d -dimensional depolarizing channel is a special case of a d -dimensional Pauli channel, with probability parameters

$$q_{mn} = \begin{cases} 1 - p + \frac{p^2}{d}, & m = n = 0 \\ \frac{p^2}{d}, & \text{otherwise.} \end{cases} \quad (35)$$

for the noise parameter p , with $0 \leq p \leq 1$, and $m, n = 0, \dots, d-1$.

In the following Lemma we make the statement that the von Neumann entropy of a state that was sent through the two-sided depolarizing channel is independent of any local unitary transformations that were performed before the action of the channel.

Lemma 4. Let $\Lambda_{ab}^{\text{dep}}$ denote a two-sided d -dimensional depolarizing channel. For a state ρ and bilateral unitary operator $U_a \otimes U_b$, we have

$$S\left(\Lambda_{ab}^{\text{dep}}\left((U_a \otimes U_b) \rho (U_a^\dagger \otimes U_b^\dagger)\right)\right) = S(\Lambda_{ab}^{\text{dep}}(\rho)). \quad (36)$$

Proof: Considering Λ_a^{dep} and Λ_b^{dep} to be the d -dimensional depolarizing channels that act on Alice's and Bob's system, respectively, it is straightforward to verify that

$$\Lambda_a^{\text{dep}}(\lambda_i) = (1-p)\lambda_i, \quad (37)$$

(where λ_i are as before the generators of $SU(d)$), and analogously for Bob's system.

Using the decomposition (5) for ρ and the following relation (proved in the Appendix A):

$$\Lambda_a^{\text{dep}}(U_a \lambda_i U_a^\dagger) = (1-p)U_a \lambda_i U_a^\dagger, \quad (38)$$

it is then easy to prove the following covariance property of the channel:

$$\Lambda_{ab}^{\text{dep}}\left((U_a \otimes U_b) \rho (U_a^\dagger \otimes U_b^\dagger)\right) = (U_a \otimes U_b) \left[\Lambda_{ab}^{\text{dep}}(\rho)\right] (U_a^\dagger \otimes U_b^\dagger). \quad (39)$$

Since the von Neumann entropy is invariant under unitary transformations, the proof of Lemma 4 is complete. \square

As a consequence of Lemma 4 we can conclude that for a two-sided d -dimensional depolarizing channel the entropy for a given initial state ρ is independent of the unitary encoding, namely

$$S\left(\Lambda_{ab}^{\text{dep}}((U \otimes \mathbb{1})\rho(U^\dagger \otimes \mathbb{1}))\right) = S\left(\Lambda_{ab}^{\text{dep}}(\rho)\right). \quad (40)$$

Therefore, (17) holds and, according to (23), the super dense coding capacity for a given general resource state ρ , with a two-sided d -dimensional depolarizing channel is given by

$$\begin{aligned} C^{\text{two-sided dep}_d}(\rho) &= S\left(\frac{\mathbb{1}}{d} \otimes \Lambda_b^{\text{dep}}(\rho_b)\right) - S\left(\Lambda_{ab}^{\text{dep}}(\rho)\right) \\ &= \log d + S\left(\Lambda_b^{\text{dep}}(\rho_b)\right) - S\left(\Lambda_{ab}^{\text{dep}}(\rho)\right). \end{aligned} \quad (41)$$

Notice that since Lemma 4 holds for any local unitary $U_a \otimes U_b$, the capacity (41) depends only on the degree of entanglement of the input state ρ . In other words, all input states with the same degree of entanglement have the same super dense coding capacity.

Comparing the above expression (41) with the one for the noiseless case, given by $C = \log d + S(\rho_b) - S(\rho)$, one realizes that in the case of two-sided noise the channel that affects Bob's subsystem enters twice, both in the von Neumann entropies for the local and the global density matrix.

4.1. Super dense coding capacity and optimal initial state

In (41) we obtained the super dense coding capacity of an arbitrary given initial resource state ρ for the *two-sided* d -dimensional depolarizing channel. In this subsection we perform the optimization of the super dense coding capacity over the initial state of two qubits for the *two-sided* 2-dimensional depolarizing channel. Thus, we derive the optimal value of the super dense coding capacity, if Alice and Bob have a depolarizing channel available for the transfer of 2-dimensional quantum states and can choose the initial resource state.

A pure state of two qubits $|\vartheta_\alpha\rangle$ can be written in the Schmidt bases $\{|u_i\rangle\}, \{|v_i\rangle\}$ as $|\vartheta_\alpha\rangle = \sqrt{1-\alpha}|u_1v_1\rangle + \sqrt{\alpha}|u_2v_2\rangle$ with $0 \leq \alpha \leq 1/2$. Two local unitaries V_a and V_b convert the computational bases to the Schmidt bases. Therefore, $|\vartheta_\alpha\rangle$ in computational bases can be written as $|\vartheta_\alpha\rangle = V_a \otimes V_b (\sqrt{1-\alpha}|00\rangle + \sqrt{\alpha}|11\rangle)$. In (36) we showed that the output von Neumann entropy of the two-sided depolarizing channel is invariant under previous local unitary transformations. Therefore $|\vartheta_\alpha\rangle$ and $|\varphi_\alpha\rangle = \sqrt{1-\alpha}|00\rangle + \sqrt{\alpha}|11\rangle$ lead to the same dense coding capacity. We can thus parametrize a pure initial state as a function of a single real parameter, namely as the state $|\varphi_\alpha\rangle$, and follow the approach

of Ref. [10]. The super dense coding capacity (41) of a pure state of two qubits as a function of α and the noise parameter p is given by

$$C_{\alpha}^{\text{two-sided dep}_2}(|\varphi_{\alpha}\rangle\langle\varphi_{\alpha}|) = 1 - \xi_1 \log \xi_1 - \xi_2 \log \xi_2 \\ + \gamma_1 \log \gamma_1 + \gamma_2 \log \gamma_2 + 2\gamma_3 \log \gamma_3 , \quad (42)$$

where γ_i (with $i = 1, 2, 3, 4$) are the eigenvalues of $\Lambda_{ab}^{\text{dep}}(|\varphi_{\alpha}\rangle\langle\varphi_{\alpha}|)$ and ξ_s (with $s = 1, 2$) are the eigenvalues of $\Lambda_b^{\text{dep}}(\rho_{b,\alpha})$, where $\rho_{b,\alpha} = \text{tr}_a(|\varphi_{\alpha}\rangle\langle\varphi_{\alpha}|)$. The eigenvalues γ_i and ξ_s are explicitly given by

$$\gamma_{1,2} = \frac{1}{2} \left(1 - p(1 - \frac{p}{2}) \pm (1 - p) \sqrt{1 - 4p\alpha(2 - p)(1 - \alpha)} \right) , \\ \gamma_3 = \gamma_4 = \frac{p}{2} (1 - \frac{p}{2}) , \\ \xi_1 = \alpha - p\alpha + \frac{p}{2} , \\ \xi_2 = 1 - \alpha + p\alpha - \frac{p}{2} . \quad (43)$$

We can now maximize expression (42) over the variable α , for a given noise parameter p , and find interesting results. They are illustrated in Figure 3, where we plot the superdense coding capacity in (42) as a function of the noise parameter p , for various values α . We find that there is a threshold value $p_t \approx 0.345$, where two curves cross each other: for $0 \leq p \leq 0.345$ the value $\alpha = 1/2$ leads to the highest super dense coding capacity, i.e. the optimal initial resource state is a Bell state. For $p \geq 0.345$, the optimal choice is $\alpha = 0$, i.e. product states are best for dense coding. As shown graphically in the close-up of Figure 3, the curves for intermediate values of α are always lower than $\alpha = 1/2$ or $\alpha = 0$. In order to prove this claim, we also evaluated $C_{\alpha=1/2}^{\text{two-sided dep}_2} - C_{\alpha}^{\text{two-sided dep}_2}$ in the range of $0 \leq p \leq 0.345$ and $C_{\alpha=0}^{\text{two-sided dep}_2} - C_{\alpha}^{\text{two-sided dep}_2}$ in the range of $0.345 \leq p \leq 1$ as functions of the parameters α and p . We found that these two functions are positive or zero. Thus, for pure initial states it is always best to either use maximally entangled states or product states, depending on the noise level.

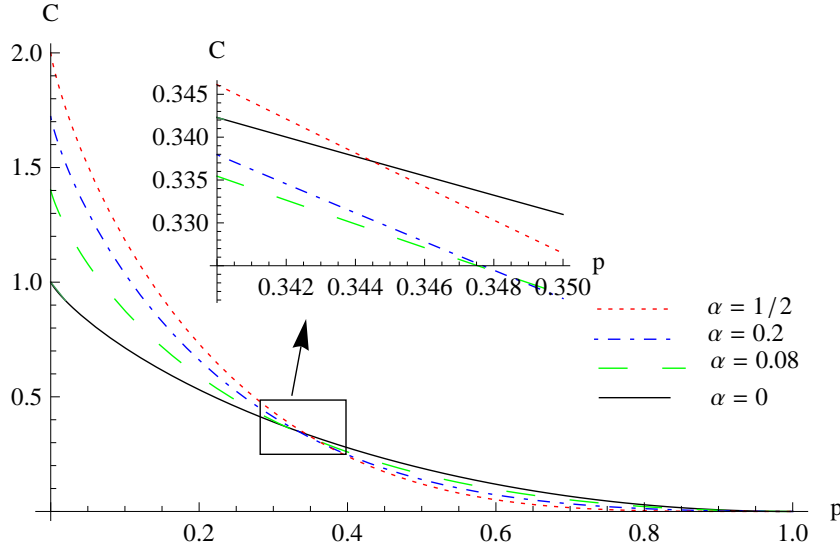


Figure 3. The super dense coding capacity for the two-sided depolarizing channel in 2 dimensions, $C_{\alpha}^{\text{two-sided dep}_2}$, as function of the noise parameter p , for $\alpha = 0$, $\alpha = 0.08$, $\alpha = 0.2$ and $\alpha = 1/2$. For the definition of α see main text. For $0 \leq p \leq 0.345$ a Bell state, i.e. $\alpha = 1/2$, leads to the optimal capacity, while for $0.345 \leq p \leq 1$ the optimal initial state is a product state ($\alpha = 0$).

In the following we call the super dense coding capacity of an initial Bell state $|\varphi_{1/2}\rangle$ in the presence of a *two-sided* 2-dimensional depolarizing channel $C_{\text{Bell}}^{\text{two-sided dep}_2}$. Using (42) with $\alpha = 1/2$, this capacity is given by

$$C_{\text{Bell}}^{\text{two-sided dep}_2} = 2 + \frac{1 + 3(1-p)^2}{4} \log \frac{1 + 3(1-p)^2}{4} + 3 \frac{1 - (1-p)^2}{4} \log \frac{1 - (1-p)^2}{4}. \quad (44)$$

The super dense coding capacity with an initial product state $|\varphi_0\rangle$ in the presence of a *two-sided* 2-dimensional depolarizing channel is denoted in the following as $C^{\text{ch dep}_2}$. From (42) with $\alpha = 0$ it follows that

$$C^{\text{ch dep}_2} = 1 + \frac{p}{2} \log \frac{p}{2} + \frac{2-p}{2} \log \frac{2-p}{2}. \quad (45)$$

Note that (45) is identical to the classical channel capacity of the depolarizing channel for qubits [11].

We now show that using mixed initial states as a resource cannot increase the super dense coding capacity, i.e. $|\varphi_{1/2}\rangle$ and $|\varphi_0\rangle$ are the optimal input states for the range of noise parameter $0 \leq p \leq 0.345$ and $0.345 \leq p \leq 1$, respectively. To show this claim we first write the super dense coding capacity (41) in the form of the relative entropy

$$C^{\text{two-sided dep}_d}(\rho) = S(\Lambda_{ab}(\rho) \| \frac{\mathbb{1}}{d} \otimes \Lambda_b(\rho_b)). \quad (46)$$

Since any mixed state can be written as a convex combination of pure states ρ_k , i.e. $\rho_{mix} = \sum_k p_k \rho_k$, and $\rho_{b,mix} = \text{tr}_a(\rho_{mix}) = \sum_k p_k \rho_{b,k}$, we can write

$$\begin{aligned} C_{\rho_{mix}} &= S(\Lambda_{ab}(\rho_{mix}) \| \tilde{\rho}) = S(\Lambda_{ab}(\rho_{mix}) \| \frac{\mathbb{1}}{d} \otimes \Lambda_b(\rho_{b,mix})) \\ &= S(\sum_k p_k \Lambda_{ab}(\rho_k) \| \sum_k p_k \frac{\mathbb{1}}{d} \otimes \Lambda_b(\rho_{b,k})) \\ &\leq \sum_k p_k S(\Lambda_{ab}(\rho_k) \| \frac{\mathbb{1}}{d} \otimes \Lambda_b(\rho_{b,k})) . \end{aligned} \quad (47)$$

In the above inequality we have used the subadditivity of the relative entropy, i.e. $S(\sum_i p_i r_i \| \sum_i q_i s_i) \leq \sum_i p_i S(r_i \| s_i) + H(p_i \| q_i)$, where $H(\cdot \| \cdot)$ is the Shannon relative entropy, defined as $H(p_i \| q_i) = \sum_i p_i \log \frac{p_i}{q_i}$ [13]. We showed before that the super dense coding capacity of a pure state for $0 \leq p \leq 0.345$ is upper bounded by the super dense coding capacity of a Bell state $|\varphi_{1/2}\rangle$, and for $0.345 \leq p \leq 1$ it is upper bounded by the product state $|\varphi_0\rangle$. Remembering that ρ_k is pure, and using (46), we find that for $0 \leq p \leq 0.345$

$$C_{\rho_{mix}} \leq \sum_k p_k S(\Lambda_{ab}(\rho_k) \| \frac{\mathbb{1}}{d} \otimes \Lambda_b(\rho_{b,k})) \leq C^{\text{two-sided dep}_2} , \quad (48)$$

and for $0.345 \leq p \leq 1$

$$C_{\rho_{mix}} \leq \sum_k p_k S(\Lambda_{ab}(\rho_k) \| \frac{\mathbb{1}}{d} \otimes \Lambda_b(\rho_{b,k})) \leq C^{\text{ch dep}_2} , \quad (49)$$

which proves our claim.

It is interesting to note that the optimal capacity for the two-sided qubit depolarizing channel is a non-differentiable function of the noise parameter p , and that the optimal states are either maximally entangled or separable. In other words, there is a transition in the entanglement of the optimal input states at the particular threshold value of the noise parameter $p_t \approx 0.345$. Notice that a similar transition behavior in the entanglement of the optimal input states for transmission of classical information was found also for the qubit depolarizing channel with correlated noise [16]. It is interesting that in the present context the transition behavior arises in a memoryless channel and is not related to correlations introduced via the noise process.

5. Super dense coding capacity versus channel capacity

In this section, we consider the question of whether or not it is reasonable in the presence of noise to use the super dense coding protocol for the transmission of classical information? To answer this question, we provide a comparison between the classical capacity of a 2-dimensional depolarizing channel and the super dense coding capacities

of a *one-sided* and *two-sided* 2-dimensional depolarizing channel, for the resource of an initial Bell state. Since the depolarizing channel is a special form of a Pauli channel, according to (31) the super dense coding capacity for a *one-sided* 2-dimensional depolarizing channel for an initially shared Bell state is

$$C^{\text{one-sided dep}_2} = 2 + \frac{4-3p}{4} \log \frac{4-3p}{4} + 3\frac{p}{4} \log \frac{p}{4}. \quad (50)$$

The super dense coding capacity for a *two-sided* 2-dimensional depolarizing channel with a Bell state as resource is given in (44). The classical capacity $C^{\text{ch dep}_2}$ of the 2-dimensional depolarizing channel is achieved by an ensemble of pure states belonging to an orthonormal basis, say $\{|0\rangle, |1\rangle\}$ at the channel input, with equal probability $\frac{1}{2}$ and performing a complete von Neumann measurement in the same basis over the channel output [11]. Its expression is given explicitly in (45).

In Figure 4, we plot $C^{\text{one-sided dep}_2}$, $C^{\text{two-sided dep}_2}$, $C^{\text{ch dep}_2}$, and $C = 1$ in terms of the noise parameter p . As we expect, the first three capacities $C^{\text{one-sided dep}_2}$, $C^{\text{two-sided dep}_2}$ and $C^{\text{ch dep}_2}$ decrease as the noise increases. As expected, the super dense coding capacity of a *one-sided* 2-dimensional depolarizing channel $C^{\text{one-sided dep}_2}$ is greater than the classical capacity $C^{\text{ch dep}_2}$ for all values of p , as the additional resource of entanglement is used in dense coding. The comparison between $C^{\text{two-sided dep}_2}$ and $C^{\text{ch dep}_2}$ illustrates that for $0.345 \leq p \leq 1$ the 2-dimensional depolarizing channel capacity is greater than the super dense coding capacity for a *two-sided* 2-dimensional depolarizing channel. This suggests that for $0.345 \leq p \leq 1$ Alice and Bob do not win by sending classical information via a super dense coding protocol with unitary encoding. For this regime, the noise degrades the entanglement too much to be useful. Now we can answer the question posed at the beginning of this section: super dense coding is not always a useful scheme for sending classical information in the presence of noise.

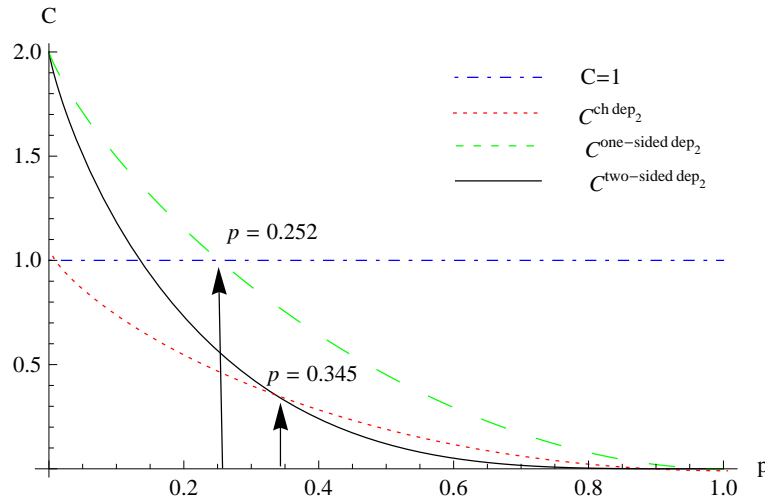


Figure 4. The classical capacity $C^{\text{ch dep}_2}$ of the 2-dimensional depolarizing channel and the super dense coding capacities for an initial Bell state in the presence of a *one-sided* and *two-sided* 2-dimensional depolarizing channel, $C^{\text{one-sided dep}_2}$ and $C^{\text{two-sided dep}_2}$, respectively, as functions of the noise parameter p .

We notice also that $C^{\text{one-sided dep}_2}$ corresponds to the entanglement assisted capacity for the depolarizing channel [15]. According to (50) for $p = 0.252$ the super dense coding capacity for an initial Bell state via the *one-sided* 2-dimensional depolarizing channel is equal to *one*. The maximum information that can be transmitted by two-dimensional systems without any source of entangled states is $C = 1$. That is, for $p = 0.252$ the super dense coding capacity reaches the classical limit, as can be seen in Figure 4. It was shown in [14] that the classical limit of the quantum teleportation protocol, when using a Bell state and distributing one subsystem of it via a depolarizing channel, is reached at $p = 1/3$. In the absence of noise, quantum teleportation and super dense coding are two equivalent protocols [12]. According to our results this is not true in the presence of noise, as we have shown explicitly for the depolarizing channel: here, the quantum/classical boundary for super dense coding occurs at a different noise value than for quantum teleportation.

We point out that the expression (31) for the dense coding capacity of a Bell state provides a lower bound to the entanglement-assisted capacity of a general Pauli channel.

6. Non-unitary encoding for the d -dimensional Pauli channel

So far, we assumed that the encoding in the super dense coding protocol is unitary. In this section we consider the possibility of performing non-unitary encoding on the initial state and discuss explicitly the case of the depolarizing channel. Let us consider Γ_i to be a completely positive trace preserving (CPTP) map. Alice applies the map Γ_i on her side of the shared state ρ , thereby encoding ρ as $\rho^i = [\Gamma_i \otimes \mathbb{I}](\rho) := \Gamma_i(\rho)$. The super dense coding protocol with non-unitary encoding for noiseless channels has been discussed by M. Horodecki et al. [17], M. Horodecki and Piani [5], and Winter [18]. In this section we introduce an upper bound on the Holevo quantity for a *two-sided* d -dimensional Pauli channel, and show that this upper bound is reachable by a pre-processing before unitary encoding. Our arguments follow a similar line as in [5], where non-unitary encoding was studied for the case of noiseless channels.

Lemma 5. Let $\chi = S(\sum_i p_i \Lambda_{ab}^P(\rho_i)) - \sum_i p_i S(\Lambda_{ab}^P(\rho_i))$ be the Holevo quantity with $\rho_i = \Gamma_i(\rho)$ and let $\Lambda_{ab}^P(\rho)$ be a general *two-sided* d -dimensional Pauli channel defined via

$$\Lambda_{ab}^P(\rho) = \sum_{m,n,\tilde{m},\tilde{n}=0}^{d-1} q_{mn\tilde{m}\tilde{n}} (V_{mn} \otimes V_{\tilde{m}\tilde{n}})(\rho) (V_{mn}^\dagger \otimes V_{\tilde{m}\tilde{n}}^\dagger) \quad (51)$$

with $\sum_{m,n,\tilde{m},\tilde{n}=0}^{d-1} q_{mn\tilde{m}\tilde{n}} = 1$. Let $\Gamma_M(\cdot) := [\Gamma_M \otimes \mathbb{I}](\cdot)$ be the map that minimizes the von Neumann entropy after application of this map and the channel Λ_{ab} to the initial state ρ , i.e. Γ_M minimizes the expression $S(\Lambda_{ab}^P(\Gamma_M(\rho)))$. Then, the Holevo quantity χ is upper bounded by

$$\chi \leq \log d + S(\Lambda_b^P(\rho^B)) - S(\Lambda_{ab}^P(\Gamma_M(\rho))). \quad (52)$$

Proof: $\Gamma_M(\cdot)$ is a map that leads to the minimum of the entropy after applying it and the channel to the initial state ρ . Therefore,

$$\begin{aligned}\chi &= S\left(\sum_i p_i \Lambda_{ab}^P(\rho^i)\right) - \sum_i p_i S(\Lambda_{ab}^P(\rho^i)) \\ &\leq S\left(\sum_i p_i \Lambda_{ab}^P(\rho^i)\right) - S(\Lambda_{ab}^P(\Gamma_M(\rho))).\end{aligned}$$

Since the von Neumann entropy is subadditive and since the maximum entropy of a d -dimensional system is $\log d$, we have

$$\chi \leq \log d + S\left(\text{tr}_a \left(\sum_i p_i \Lambda_{ab}^P(\rho^i)\right)\right) - S(\Lambda_{ab}^P(\Gamma_M(\rho))).$$

Now, since $\text{tr}_a \sum_i p_i \Lambda_{ab}^P(\rho^i) = \Lambda_b^P(\rho_b)$ it follows that

$$\chi \leq \log d + S(\Lambda_b^P(\rho_b)) - S(\Lambda_{ab}^P(\Gamma_M(\rho))).$$

□

If the upper bound in (52) is achievable, then it is equal to the super dense coding capacity. We consider the ensemble $\{\tilde{p}_i, \tilde{\Gamma}_i(\rho)\}$ with $\tilde{p}_i = \frac{1}{d^2}$ and $\tilde{\Gamma}_i(\rho) = (V_i \otimes \mathbb{1})\Gamma_M(\rho)(V_i^\dagger \otimes \mathbb{1})$, where V_i is defined in (6). We will show in the following that this ensemble achieves the upper bound in (52). In other words, the optimal encoding consists of a fixed pre-processing with Γ_M and a subsequent unitary encoding. This is analogous to the case of noiseless channels, for which the same statement was shown in [5]. The Holevo quantity of the ensemble $\{\tilde{p}_i, \tilde{\Gamma}_i(\rho)\}$ is

$$\tilde{\chi} = S\left(\sum_i \frac{1}{d^2} \Lambda_{ab}^P(\tilde{\Gamma}_i(\rho))\right) - \sum_i \frac{1}{d^2} S\left[\Lambda_{ab}^P(\tilde{\Gamma}_i(\rho))\right]. \quad (53)$$

By using (9) and noting that Γ_M acts only on Alice's side, and by using Lemma 1-a), we find that the average of $\Lambda_{ab}^P(\tilde{\Gamma}_i(\rho))$, i.e. the argument in the first term on the RHS of (53), is given by

$$\sum_i \frac{1}{d^2} \Lambda_{ab}^P(\tilde{\Gamma}_i(\rho)) = \frac{\mathbb{1}}{d} \otimes \Lambda_b^P(\rho_b). \quad (54)$$

Furthermore, the second term on the RHS of (53) is given by

$$\begin{aligned}\sum_i \frac{1}{d^2} S\left(\Lambda_{ab}^P(\tilde{\Gamma}_i(\rho))\right) &= \sum_i \frac{1}{d^2} S\left(\Lambda_{ab}^P\left((V_i \otimes \mathbb{1})\Gamma_M(\rho)(V_i^\dagger \otimes \mathbb{1})\right)\right) \\ &= \frac{1}{d^2} \sum_i S\left((V_i \otimes \mathbb{1}) \left[\sum_{m,n,\tilde{m},\tilde{n}=0}^{d-1} q_{mn\tilde{m}\tilde{n}} (V_{mn} \otimes V_{\tilde{m}\tilde{n}}) \Gamma_M(\rho) (V_{mn}^\dagger \otimes V_{\tilde{m}\tilde{n}}^\dagger) \right]\right).\end{aligned}$$

$$\begin{aligned}
& \cdot (V_i^\dagger \otimes \mathbb{I})) \\
& = \frac{1}{d^2} \sum_i S [\Lambda_{ab}^P(\Gamma_M(\rho))] = S [\Lambda_{ab}^P(\Gamma_M(\rho))]
\end{aligned} \tag{55}$$

where in the second line of the above equations we have inserted the action of the channel, defined in (51), and we have used (28), from which it follows that V_i and V_{mn} commute up to a phase.

Inserting (54) and (55) into (53), one finds that the Holevo quantity $\tilde{\chi}$ is equal to the upper bound given in (52). Consequently, the super dense coding capacity with non-unitary encoding is

$$C = \log d + S(\Lambda_b^P(\rho_b)) - S[\Lambda_{ab}^P(\Gamma_M(\rho))] . \tag{56}$$

Thus, we have shown above for the case of a d -dimensional Pauli channel that applying the appropriate pre-processing Γ_M on the initial state ρ before the unitary encoding $\{V_i\}$ may increase the super dense coding capacity, with respect to only using unitary encoding. Our results derived in section 4 provide an example where pre-processing indeed leads to an improvement: Consider the case of a *two-sided* 2-dimensional depolarizing channel for an initial Bell state with a noise parameter in the range $0.345 \leq p \leq 1$, see Figure 3. To reach the optimal super dense coding capacity in this case, Alice applies a measurement as a pre-processing, projecting the Bell state onto $|00\rangle$ or $|11\rangle$; afterwards she applies the unitary encoding. As we showed above, the super dense coding capacity for product states is equal to the capacity of the depolarizing channel, given in (45). Thus, in this case we reach a higher super dense coding capacity than without pre-processing. The effect of pre-processing is illustrated in Figure 5, which is an excerpt of Figure 4.

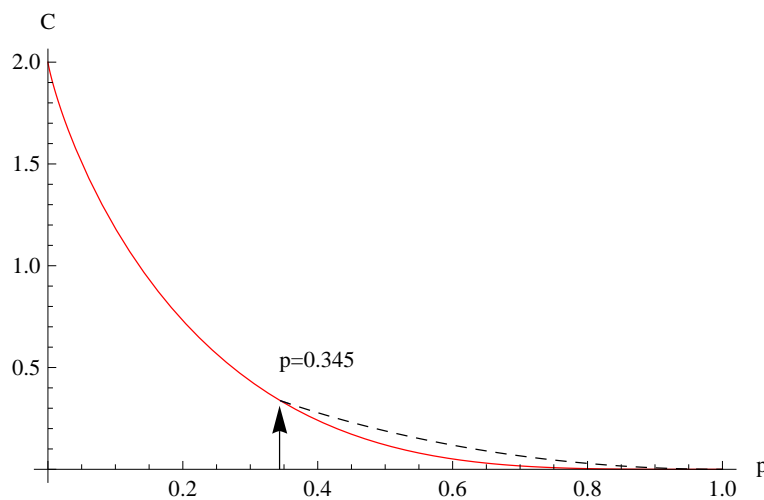


Figure 5. The solid curve is the optimal super dense coding capacity with a Bell state in the presence of a *two-sided* 2-dimensional depolarizing channel. The dashed line shows the improved super dense coding capacity by using a pre-processing on the Bell state in the range of $0.345 \leq p \leq 1$.

7. Conclusions

In conclusion, we investigated the bipartite super dense coding protocol in the presence of a unital noisy channel, which acts either only on Alice's subsystem after encoding (*one-sided* channel) or both on Alice's and Bob's subsystems (*two-sided* channel). For those cases where the von Neumann entropy fulfills a specific condition, we derived the super dense coding capacity. We showed that a *one-sided* d -dimensional Pauli channel for the resource of Bell and Werner states fulfills the above mentioned condition on the von Neumann entropy. Our condition on the von Neumann entropy is also satisfied for a *two-sided* d -dimensional depolarizing channel. For these examples, we derived the explicit optimal super dense coding capacity, as a function of the initial resource state. When the initial state can be chosen, we found for the case of a *two-sided* 2-dimensional depolarizing channel that the optimal initial resource state is either a Bell state or a product state, depending on the value of noise parameter.

We also compared the classical capacity of the 2-dimensional depolarizing channel to the super dense coding capacities for an initial Bell state with a *one-sided* and *two-sided* 2-dimensional depolarizing channel. Our results showed that Alice and Bob may not win by sending classical information via a super dense coding protocol with unitary encoding, if there is too much noise. Comparing the critical noise parameters for the quantum/classical boundary, we found that in the scenario of the depolarizing channel the protocols quantum teleportation and super dense coding are not equivalent, in the sense that they do not have the same critical noise parameter.

Finally, we discussed the super dense coding capacity with non-unitary encoding for a *two-sided* d -dimensional Pauli channel. We showed that the optimal strategy is to apply a pre-processing before the unitary encoding. We gave an example of super dense coding for an initial Bell state and a *two-sided* 2-dimensional depolarizing channel where pre-processing increases the super dense coding capacity, as compared to only unitary encoding.

There are several open questions: how can the super dense coding capacity be determined for other channels and states than the ones that fulfil the specific entropy condition? What is the influence of correlated noisy channels? How does noise affect the multipartite super dense coding scenario? These topics will be addressed in future work.

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Appendix A.

We give here a proof for (38). We expand $U\lambda_i U^\dagger$ in terms of $\{V_{mn}\}$. By using the fact that λ_i is traceless, we have

$$\Lambda_a^{dep}(U\lambda_i U^\dagger) = \Lambda_a^{dep} \left(\sum_{m,n \neq (0,0)}^{d-1} \gamma_{mn} V_{mn} \right).$$

Here, $\Lambda_a^{dep}(\cdot)$ is a linear map that is given by $\Lambda_a^{dep}(\cdot) = \sum_{\tilde{m}, \tilde{n}=0}^{d-1} q_{\tilde{m}\tilde{n}} V_{\tilde{m}\tilde{n}}(\cdot) V_{\tilde{m}\tilde{n}}^\dagger$. Then we can write

$$\begin{aligned} \Lambda_a^{dep}(U\lambda_i U^\dagger) &= \sum_{m,n \neq (0,0)}^{d-1} \gamma_{mn} \Lambda_a^{dep}(V_{mn}) \\ &= \sum_{\tilde{m}, \tilde{n}=0}^{d-1} \sum_{m,n \neq (0,0)}^{d-1} \gamma_{mn} q_{\tilde{m}\tilde{n}} V_{\tilde{m}\tilde{n}} V_{mn} V_{\tilde{m}\tilde{n}}^\dagger. \end{aligned}$$

By using (28) and unitarity of V_{mn} , we have

$$\begin{aligned} \Lambda_a^{dep}(U\lambda_i U^\dagger) &= \sum_{\tilde{m}, \tilde{n}=0}^{d-1} \sum_{m,n \neq (0,0)}^{d-1} \gamma_{mn} q_{\tilde{m}\tilde{n}} \exp \left(\frac{2i\pi(n\tilde{m} - \tilde{n}m)}{d} \right) V_{mn} \\ &= \sum_{m,n \neq (0,0)}^{d-1} \gamma_{mn} V_{mn} \sum_{\tilde{m}, \tilde{n}=0}^{d-1} q_{\tilde{m}\tilde{n}} \exp \left(\frac{2i\pi(n\tilde{m} - \tilde{n}m)}{d} \right). \end{aligned}$$

For $q_{\tilde{m}\tilde{n}}$ we replace the expression of (34) and we then have

$$\begin{aligned} \Lambda_a^{dep}(U\lambda_i U^\dagger) &= \sum_{m,n \neq (0,0)}^{d-1} \gamma_{mn} V_{mn} \left(1 - p + \underbrace{\frac{p}{d^2} \sum_{\tilde{m}, \tilde{n}=0}^{d-1} \exp \left(\frac{2i\pi(n\tilde{m} - \tilde{n}m)}{d} \right)}_{\delta_{0,m} \delta_{0,n}} \right) \\ &= (1-p) \sum_{m,n \neq (0,0)}^{d-1} \gamma_{mn} V_{mn} = (1-p) U\lambda_i U^\dagger \end{aligned}$$

Therefore, (38) is proved.

Appendix B.

In Lemma 3 we need to prove that $\rho_{00}(U^\dagger V_{mn}^\dagger V_{\tilde{m}\tilde{n}} U \otimes \mathbb{1}) \rho_{00} = 0$. We here show that $\langle \psi_{00} | (U^\dagger V_{mn}^\dagger V_{\tilde{m}\tilde{n}} U \otimes \mathbb{1}) | \psi_{00} \rangle = 0$, from which the previous statement follows. Due to (27) for $m \neq \tilde{m}$ and $n \neq \tilde{n}$ the expression $V_{mn}^\dagger V_{\tilde{m}\tilde{n}}$ is traceless and $\{V_{jk}\}_{j,k=0}^{d-1}$ form a complete

set. We can thus expand $V_{mn}^\dagger V_{\tilde{m}\tilde{n}} = \sum_{(j,k) \neq (0,0)} \beta_{jk} V_{jk}$ with expansion coefficients β_{jk} . Therefore,

$$\begin{aligned}
\langle \psi_{00} | (U^\dagger V_{mn}^\dagger V_{\tilde{m}\tilde{n}} U \otimes \mathbb{1}) | \psi_{00} \rangle &= \sum_{(j,k) \neq (0,0)} \beta_{jk} \langle \psi_{00} | (U^\dagger V_{jk} U \otimes \mathbb{1}) | \psi_{00} \rangle \\
&= \frac{1}{d} \sum_{(j,k) \neq (0,0)} \sum_{m,n=0}^{d-1} \beta_{jk} \langle mm | (U^\dagger V_{jk} U \otimes \mathbb{1}) | nn \rangle \\
&= \frac{1}{d} \sum_{(j,k) \neq (0,0)} \sum_{m,n=0}^{d-1} \beta_{jk} \langle m | U^\dagger V_{jk} U | n \rangle \langle m | n \rangle \\
&= \frac{1}{d} \sum_{(j,k) \neq (0,0)} \beta_{jk} \text{tr}[U^\dagger V_{jk} U] = \frac{1}{d} \sum_{(j,k) \neq (0,0)} \beta_{jk} \text{tr}[V_{jk}] = 0.
\end{aligned}$$

Since $\rho_{00} = |\psi_{00}\rangle\langle\psi_{00}|$, we arrive at

$$\rho_{00} (U^\dagger V_{mn}^\dagger V_{\tilde{m}\tilde{n}} U \otimes \mathbb{1}) \rho_{00} = 0, \quad (\text{B.1})$$

which completes the proof.

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